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Power Series of the Free Field as Operator-Valued Functionals on Spaces of Type S†

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Abstract

Infinite series of Wick powers of the free, massive Bose field are analysed in terms of test function spaces of type S for arbitrary space dimension. By direct estimates of the smeared phase space integrals sufficiency conditions for the existence of the vacuum expectation values are derived. These conditions are shown to be precise. The field-operators are defined on a dense invariant domain in Fock space, where they satisfy the Wightman axioms with the possible exception of locality. Localisable and non-localisable fields are dealt within the same frame. The behaviour of spectral functions and the strength of singularities are discussed.

1. *Introduction*

There are not many constructive examples of quantised fields which satisfy the Wightman axioms. A thoroughly investigated class are the Wick powers $:\phi^r:(g)$ of the Klein-Gordon field $\phi(g)$ (Wightman & Garding, 1964). These powers $:\phi^r:(g)$ are obtained by shifting all annihilation operators in $\prod_{i=1}^r \phi(g_i)$ to the right, extending this multilinear functional to functions $h(x_1,\ldots,x_r)$, and performing the limit $h(x_1,\ldots,x_r) \to \prod_{i=2}^r \delta(x_1 - x_i)g(x_1)$. If the spatial part of the space-time continuum has a dimension greater than one, the Wick powers behave at the light-cone like negative powers (i.e., their vacuum expectation values have singularities of this kind), and all Wick polynomials of arbitrary degree can be defined as operator-valued functionals in the Schwartz space $\mathscr{S}'$. The straightforward generalisations, the infinite series of Wick powers

$$\chi(g) = \sum_{r=0}^{\infty} \frac{d_r}{r!} :\phi^r:(g) \tag{1.1}$$

however, have essential or even nonlocalisable singularities and can thus not be elements of $\mathscr{S}'$. Only in the two-dimensional space-time continuum

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the possibility of power series in $\mathscr{S}'$ is not excluded, since in this case the Wick powers have only logarithmic singularities. The requirement of not so customary test function spaces seems to have prevented the systematic discussion of the fields (1.1) in higher dimensions hitherto, although already the simplest field-theoretic models demand for their solution transcendental functions of the free field. Since there is also an increasing interest in non-polynomial Lagrangian theories (Salam, 1969a), a detailed discussion of the infinite series $\chi(g)$ in the spirit of Wightman should not be postponed.

The lines one has to follow are obvious: one has to restrict the functional domain of definition of the fields to subspaces of $\mathscr{S}$. The first step in that direction was made long ago (Güttinger, 1958). There it was proposed to take the test functions in position space out of $\mathscr{Z}$ (the space of analytic functions) and to accept the resulting nonlocal features as an inherent structure of nonrenormalisable theories. It was Schroer (1964) who at first observed that there is a large class of nonrenormalisable fields which are still localisable (i.e., they can be smeared with test functions of compact support in position space). A general functional-theoretic frame for all localisable fields was given by Jaffe (1966, 1967, 1968). The test function spaces he introduced are characterised by an indicator function, i.e., by infinitely many parameters. For our investigation of the power series we found it convenient to use a family of test function spaces which are indexed by one or two constants only, since the growth of power series is commonly described by the two numbers order and type also. The appropriate spaces $\mathscr{S}^{\alpha,A}$ (for position space) and $\mathscr{S}_{\alpha,A}$ (for momentum space) were introduced by Gelfand & Schilov (1962) and belong to the so-called spaces of type S. The family $\mathscr{S}^{\alpha,A}$ interpolates the region between $\mathscr{S}$ and $\mathscr{Z}$, and $\mathscr{S}_{\alpha,A}$ between $\mathscr{S}$ and $\mathscr{D}$. They include also the case where the dual spaces contain nonlocalisable functions (that is, for $\alpha < 1$). This is desirable, because the set of localisable fields is too small for all applications. The basic definitions and properties of the spaces of type S are stated in Appendix 1. There are also proven two new theorems for functionals in $\mathscr{S}^{\alpha\prime}$, concerning the transformation to difference-variables and the connection with boundary values of analytic functions.

The main purpose of this paper is to determine for each $\alpha \in [0,\infty)$ the growth of the coefficients d_r in (1.1), which is admissible in order that the power series belong to $\mathscr{S}^{\alpha\prime}$. Since field-theoretic models are studied in the literature in various space dimensions, and since it does not complicate our calculations, we work with arbitrary space dimension (which is, for practical reasons, denoted by $\kappa+1$). For $\kappa = 0$ (two-dimensional space-time) the power series which belong to $\mathscr{S}'$ were investigated by Jaffe (1965a). Shifting the convergence problem of functionals to that of analytic functions it is shown (Jaffe, 1965a) that just the series of order unity have limits in $\mathscr{S}'$. We shall deal with the convergence problem without reference to analytic functions and are thus able to treat the nonlocalisable fields on the same footing as the localisable ones. We concentrate mainly on the case $\kappa \geqslant 1$, while the discussion of $\kappa = 0$ in terms of our method is outlined in Appendix 2.

In Section 2 sufficient conditions for the convergence of the two-point function are set up. It is also demonstrated that these conditions are precise within the accuracy given by the family of test function spaces $\mathscr{S}^{\alpha}$.

Section 3 shows that the aforementioned conditions also ensure the existence of the n-point functions.

The field operators are defined in Section 4 on a dense invariant domain in Fock space, where they satisfy the Wightman axioms. Power series of order smaller than two and of order two and type zero are shown to enlarge the Borchers class of the free field.

The results thus far can also be considered in another way. Given a field $\chi(g)$, the investigations of Sections 2–4 determine the maximal functional domain of definition. That this is not purely of academic interest is shown in Section 5. Once one has at hand the functional domain one is able to make statements on the growth of the spectral functions and the singularities at the light-cone. For these conclusions, which belong to the dynamics of a theory, the functional-theoretic results of Appendix 1 are needed. A look at simple models shows that the sufficiently precise characterised functional domain (one has to use an at least twice-indexed family of spaces such as $\mathscr{S}^{\alpha, A}$) may depend on the coupling constant. Thus the determination of the appropriate test function space for a given field is also part of the dynamics and should not be anticipated by the inclusion into a general set of axioms.

In Section 6 an operator analysis is outlined which may be of use for the study of models and nonpolynomial Lagrangians.

Working henceforth in $\kappa + 1$ space dimensions we develop some notation for later use.

$$x := (x^0, x^1, \ldots, x^{\kappa+1}), \qquad \mathbf{x} := (x^1, \ldots, x^{\kappa+1})$$

$$xy = x^0 y^0 - \mathbf{xy}, \qquad dx = dx^0 \cdots dx^{\kappa+1}, \qquad d\mathbf{x} = dx^1 \cdots dx^{\kappa+1}$$

$$i\Delta^{(+)}(x) = (2\pi)^{-(\kappa+1)} \int \exp(-ixp)\, \theta(p_0)\, \delta(p^2 - m^2)\, dp$$

$$g(x) = (2\pi)^{-(\kappa+3)/2} \int \exp(-ixp)\, \tilde{g}(p)\, dp$$

2. *The Two-Point Function*

For technical reasons we shall not discuss directly the convergence of the series (1.1) in some operator topology, but prefer to deal with the vacuum expectation values. The essential advantage of the latter is their translation invariance.

We start with the two-point function

$$\mathscr{W}^{(2)}(g_1, g_2) = (\Omega, \chi^+(g_1)\, \chi(g_2)\, \Omega)$$

$$= \sum_{r=0}^{\infty} \frac{|d_r|^2}{r!} \Delta^r(g_1, g_2) \tag{2.1}$$

where

$$\Delta^r(g_1, g_2) = \int \int g_1{}^*(x)\,(i\Delta^{(+)}(x-y))^r\, g_2(y)\, dx\, dy$$

$$= (2\pi)^{-(\kappa+1)(r-1)} \int \cdots \int \tilde{g}_1{}^*\left(\sum_{l=1}^{r} p_l\right) \tilde{g}_2\left(\sum_{l=1}^{r} p_l\right) \prod_{l=1}^{r} \frac{d\mathbf{p}_l}{2p_l{}^0} \quad (2.2)$$

and $p_l{}^0 = \sqrt{(\mathbf{p}_l{}^2 + m^2)}$. $\tilde{g}_{1/2}$ are the Fourier transforms of $g_{1/2}$ and Ω denotes the no-particle state. We suppose now that the test functions $\tilde{g}_i$ be in $\mathscr{S}_\alpha(\mathbf{R}^{\kappa+2})$ (cf. Gelfand & Schilow, 1962 and Appendix 1). In a fixed coordinate system it would be sufficient to require the $\tilde{g}_i$ to be in $\mathscr{S}_\alpha$ only with respect to the energy variable p_0, but it is preferable to work with Lorentz invariant test function spaces. According to (A1.1) and (A1.3) we have

$$|p^{0L} \tilde{g}_1(p) \tilde{g}_2(p)| \leqslant C A^L L^{L\alpha}$$

Thus

$$|\Delta^r(g_1, g_2)| \leqslant C_1\, C_2{}^r\, A^L L^{L\alpha}\, I(r, L)$$

where we have used the abbreviation

$$I(r, L) = \int \cdots \int \left(\sum_{l=1}^{r} p_l{}^0\right)^{-L} \prod_{l=1}^{r} \frac{d\mathbf{p}_l}{2p_l{}^0} \quad (2.3)$$

Integration over the angles and transition to the variables $t_i = \sum_{l=1}^{i} p_l{}^0$, $i = 1, \ldots, r$, leads to

$$I(r, L) = C^r \int_{rm}^{\infty} \int_{(r-1)m}^{t_r - m} \cdots \int_{m}^{t_2 - m} t_r^{-L} \prod_{l=2}^{r} [(t_l - t_{l-1})^2 - m^2]^{(\kappa-1)/2}$$
$$\times (t_1{}^2 - m^2)^{(\kappa-1)/2}\, dt_r \ldots dt_1$$

Constants as C, C_1 may have different values in different lines.

In the sequel we treat only the case $\kappa \geqslant 1$, discussing the case $\kappa = 0$ in Appendix 2. Since $\prod_{l=2}^{r} (t_l - t_{l-1}) t_1$ has its maximum at $(t_r/r)^r$ for t_r fixed, we get

$$I(r, L) \leqslant C^r r^{-(\kappa-1)r} \frac{1}{(\mathrm{r}-1)!} \int_{rm}^{\infty} t_r^{-L+\kappa r-1}\, dt_r$$

In order to guarantee the existence of the integral on the right-hand side we have to require that L be an integer-valued function of r so that $L(r) > \kappa r$. Then

$$I(r, L) \leqslant C^r m^{-L} \frac{r^{r-L}}{(r-1)!\,(L-\kappa r)}. \quad (2.4)$$

We use the freedom in the choice of $L(r)$ to make the expression $A^L L^{L\alpha} I(r, L)$ as small as possible.

For $\alpha \geqslant 1$ we set

$$L(r) = \gamma r, \qquad \gamma > \kappa$$

We obtain in this way the following estimate for $\mathscr{W}^{(2)}$

$$|\mathscr{W}^{(2)}(g_1,g_2)| \leqslant C_1 \sum_{r=0}^{\infty} \frac{|d_r|^2}{r!} C_2^r A^{\gamma r} r^{(\alpha-1)r} \tag{2.5}$$

From this, sufficiency conditions for the convergence of $\mathscr{W}^{(2)}$ can be deduced.

Let be $\underline{\alpha > 1}$. If there is a $\gamma' > \kappa$ and an r_0, so that

$$|d_r|^2/r! \leqslant r^{-(\alpha-1)\gamma' r}, \qquad r > r_0 \tag{2.6}$$

then (2.5) converges for all A and $\mathscr{W}^{(2)}(g_1,g_2)$ exists for all $\tilde{g}_{1/2} \in \mathscr{S}_\alpha$. (2.6) is equivalent to the requirement that the order

$$\rho\{d_r\} = \overline{\lim_{r\to\infty}} \frac{r \log r}{-\log |d_r|/r!}$$

of the series (1.1) satisfies

$$\rho\{d_r\} < \frac{2}{1+(\alpha-1)\kappa} \tag{2.7}$$

For $\underline{\alpha = 1}$ a sufficient condition is

$$|d_r|^2/r! \leqslant \sigma(r)^r, \qquad \sigma(r) > 0, \qquad \overline{\lim_{r\to\infty}}\, \sigma(r) = 0 \tag{2.8}$$

This is equivalent to

$$\rho\{d_r\} = 2$$

combined with

$$\sigma\{d_r\} = \frac{1}{e\rho} \overline{\lim_{r\to\infty}}\, r \left|\frac{d_r}{r!}\right|^{\rho/r} = 0 \tag{2.9}$$

For $\underline{0 < \alpha < 1}$ let us choose $L(r)$ to be r^β, $1 < \beta < 1/\alpha$. The majorisation of the two-point function then reads

$$|\mathscr{W}^{(2)}(g_1,g_2)| \leqslant C_1 \sum_{r=0}^{\infty} \frac{|d_r|^2}{r!} C_2^r m^{-r^\beta} A^{r^\beta} r^{(\alpha\beta-1)r^\beta} \tag{2.10}$$

A sufficient condition for convergence is given if there is a constant β', $1 < \beta' < 1/\alpha$, and an r_0, such that

$$|d_r|^2/r! \leqslant \exp(r^{\beta'}), \qquad r > r_0 \tag{2.11}$$

The coefficients of the power series are in this case allowed to increase strongly with r.

If $\underline{\alpha = 0}$, then the sequence $A^L L^{L\alpha} I(r,L) \leqslant A^L C^r m^{-L} r^{-L}$ decreases the faster, the stronger the sequence $L(r)$ increases. Thus $\Delta^r(g_1,g_2)$ tends arbitrarily fast to zero for $r \to \infty$, i.e., it terminates at a certain r_0. There is no condition on the coefficients d_r for convergence.

By means of (A1.4) we pass to the position space, and we summarise the foregoing results.

Theorem 1

Let the spatial part of space-time have the dimension $\kappa+1$, $\kappa \geqslant 1$, sufficient for the convergence of the two-point function of the field $\chi(g)=\sum_r d_r/r!{:}\phi^r{:}(g)$ for all test functions in

the space	... is the condition
$\mathscr{S}\ (\alpha=\infty)$	There is an r_0, so that $d_r=0$ for all $r>r_0$.
$\mathscr{S}^\alpha,\ 1<\alpha<\infty$	$\rho\{d_r\}<\dfrac{2}{1+(\alpha-1)\kappa}$.
$\mathscr{S}^1$	$\rho\{d_r\}=2$ and $\sigma\{d_r\}=0$.
$\mathscr{S}^\alpha,\ 0<\alpha<1$	There is an r_0 and a β, $1<\beta<1/\alpha$, so that $\lvert d_r\rvert^2/r! \leqslant \exp(r^\beta), \quad r>r_0$
$\mathscr{S}^0=\mathscr{L}$	d_r is an arbitrary sequence.

$\rho\{d_r\}$ and $\sigma\{d_r\}$ denote the order and the type of the power series $\chi(g)$.

Through Theorem 1 we have associated a condition with each space $\mathscr{S}^\alpha$, $\alpha\in[0,\infty)$. Henceforth, we shall refer to this condition briefly as 'condition α'.

The functionals in $\mathscr{S}^{1\prime}$ are just the limiting case of localisable generalised functions. In the slightly larger space $\mathscr{S}^{1',A_0}$ are already partially non-localisable quantities, i.e., functionals which are singular in a four-dimensional region in space-time. If one deals more carefully with the constants C_1, C_2, ..., than we have done above, one derives a sufficient condition for convergence in $\mathscr{S}^{1',A_0}$.

Sufficient for the convergence of the two-point function for all test functions in $\mathscr{S}^{1,A_0}$ is $\rho\{d_r\}=2$ and $\sigma\{d_r\}<b/A_0{}^\kappa$, where

$$b=(4\pi)^{\kappa+1}/2e\omega(\kappa)\kappa^\kappa,$$

$\omega(\kappa)$ denoting the area of the unit sphere in $\kappa+1$ dimensions.

Let us now investigate the precision of condition α. For this purpose we test $\mathscr{W}^{(2)}(g_1,g_2)$ with the special function

$$\tilde{g}_{1/2}(p)=\tilde{g}_s(p)=\begin{cases}\exp\{-(a/2)\,[\lvert p^0\rvert/2+\sqrt{(\lvert\mathbf{p}\rvert^2+m^2)/2}]^{1/\alpha}\}, \ \lvert p^0\rvert\geqslant m/2;\\ \text{an arbitrary, smooth, infinitely differentiable con-}\\ \text{tinuation for } \lvert p^0\rvert\leqslant m/2\end{cases}$$

$\tilde{g}_s(p)$ satisfies relation (A1.2) and is thus in $\mathscr{S}_\alpha(\mathbf{R}^{\kappa+2})$. Observing that

$$\left[\left(\sum_{l=1}^{r}\mathbf{p}_l\right)^2+m^2\right]^{1/2}\leqslant\sum_{l=1}^{r}(\mathbf{p}_l{}^2+m^2)^{1/2}$$

we have

$$\Delta^r(g_s, g_s) \geqslant C_1\, C_2^r \int \cdots \int \exp\left[-a\left(\sum_{l=1}^{r} p_l^0\right)^{1/\alpha}\right] \prod_{l=1}^{r} \frac{d\mathbf{p}_l}{2p_l^0}$$

As is shown in Appendix 3 a somewhat complicated estimation leads to the incomplete Γ-integral

$$\Delta^r(g_s, g_s) \geqslant C_1\, C_2^r (r!)^{-\kappa}\, a^{-\kappa r \alpha} \int\limits_{a[r(m+\epsilon)]^{1/\alpha}}^{\infty} \exp(-s)\, s^{\alpha r \kappa - 1}\, ds \qquad (2.12)$$

where ϵ is arbitrary, greater than zero. An approximation formula (Magnus & Oberhettinger, 1948) gives for $\alpha \geqslant 1$ (and $a \leqslant 1$)

$$\Delta^r(g_s, g_s) \geqslant C_1\, M(a)^r\, r^{(\alpha-1)\kappa r} \qquad (2.13)$$

where $M(a) \to \infty$ for $a \to 0$.

For $0 < \alpha < 1$ we obtain

$$\Delta^r(g_s, g_s) \geqslant C_1\, \bar{M}(a)^r \exp(-b r^{1/\alpha}) \qquad (2.14)$$

where $\bar{M}(a) \to \infty$, for $a \to 0$.

From (2.13) and (2.14) follows immediately the following theorem.

Theorem 2

The two-point function (2.2) diverges for (at least) one test function in

the space	if
$\mathscr{S}$ ($\alpha = \infty$)	$d_r \neq 0$ for infinitely many r.
$\mathscr{S}^\alpha$, $1 \leqslant \alpha < \infty$	There is an M so that $M^r r^{-(\alpha-1)\kappa r} \leqslant \lvert d_r\rvert^2/r!$, for infinitely many r
$\mathscr{S}^\alpha$, $0 < \alpha < 1$	There is an M and a b, so that $M^r \exp(b r^{1/\alpha}) \leqslant \lvert d_r\rvert^2/r!$, for infinitely many r

Corollary

Condition α is precise in the sense that not all series satisfying this condition for a fixed α also possess an existing two-point function in a greater space $\mathscr{S}^{\alpha_0}$, $\alpha < \alpha_0$.

3. *The n-Point Function*

According to Jaffe (1965a) we have for the vacuum expectation values of n fields

$$\chi^{(i)}(g) = \sum_{r=0}^{\infty} \frac{d_r^{(i)}}{r!} :\phi^r:(g), \qquad i = 1, \ldots, n$$

the expansion

$$\mathscr{W}^{(n)}(g_1,\dots,g_n) = (\Omega, \chi^{(1)}(g_1)\cdots\chi^{(n)}(g_n)\,\Omega)$$
$$= \sum_{\substack{r_{ij}=\sigma \\ 1\leqslant i<j\leqslant n}}^{\infty} \frac{d_{S_1}^{(1)}\cdots d_{S_n}^{(n)}}{r_{12}!\cdots r_{n-1,n}!} \int\cdots\int (i\Delta^{(+)}(x_1-x_2))^{r_{12}}\cdots\times$$
$$\times (i\Delta^{(+)}(x_{n-1}-x_n))^{r_{n-1,n}}\, g_1(x_1)\cdots g_n(x_n)\,dx_1\cdots dx_n$$
$$=: \sum_{\mathbf{R}\geqslant\sigma} \frac{d_{S_1}^{(1)}\cdots d_{S_n}^{(n)}}{\mathbf{R}!} \Delta^{\mathbf{R}}(g_1,\dots,g_n) \tag{3.1}$$

where $r_{ij}=r_{ji}, r_{ii}=0, S_i=\sum_{j=1}^{n} r_{ij}, \mathbf{R}=(r_{12}, r_{13}\cdots r_{n-1,n})$ and $\mathbf{R}! = r_{12}!r_{13}!\cdots r_{n-1,n}!$.

Equation (3.1) presents a simplified version of a perturbation series. In contrast to ordinary perturbation theory the lines in a graphical representation of each perturbation term would stand for a $\Delta^{(+)}$-function and the summation would be taken over the number of lines (and not over the number of vertices). We now investigate the convergence of the summed-up perturbation series and for clarity combine the necessary steps into lemmata. In the sequel we shall also make use of the notations

$$R_i = \sum_{j=i+1}^{n} r_{ij} \qquad \text{and} \qquad \|\mathbf{R}\| = R = \sum_{i=1}^{n-1} R_i \tag{3.2}$$

Lemma 1

Let $g_1, \dots, g_n$ be in $\mathscr{S}^\alpha$, then

$$|\Delta^{\mathbf{R}}(g_1,\dots,g_n)| \leqslant \begin{cases} C_1 \prod_{k=1}^{n-1}\prod_{j=k+1}^{n} C_{2,k}^{r_{kj}} A_k^{r_{kj}\gamma} r_{kj}^{(\alpha-1)\gamma r_{kj}}, & \gamma>\kappa, \text{ for } 1\leqslant\alpha \\ C_1 \prod_{k=1}^{n-1}\prod_{j=k+1}^{n} C_{2,k}^{r_{kj}} A^{r_{kj}{}^\beta} r_{kj}^{(\alpha\beta-1)r_{kj}{}^\beta}, & 1<\beta<1/\alpha, \text{ for } 0<\alpha<1 \end{cases}$$

Proof: Passing to difference variables according to Theorem A.1 we get

$$\Delta^{\mathbf{R}}(g_1,\dots,g_n) = \Delta_d^{\mathbf{R}}(h)$$
$$= C_1\, C_2^R \int\cdots\int \tilde{h}(-q_1,\dots,-q_{n-1}) \prod_{1\leqslant i<j\leqslant n}\prod_{l=1}^{r_{ij}} \frac{dp_{ijl}}{2p_{ijl}^0}$$

where

$$h(\xi_1,\dots,\xi_{n-1}) = \int \prod_{j=1}^{n} g_i\left(\sum_{i=j}^{n}\xi_i\right) d\xi_n$$

and

$$q_k = \sum_{i=1}^{k} \sum_{j=k+1}^{n} \sum_{l=1}^{r_{ij}} p_{ijl}$$

Lemma A1 shows that $\tilde{h}$ is in $\mathscr{S}_\alpha$ and satisfies the inequality (A1.5). Denoting

$$H = H(L_1, \ldots, L_{n-1}) = \prod_{k=1}^{n-1} A_k^{L_k} L_k^{L_k \alpha}$$

and observing

$$q_k{}^0 \geqslant q_k^{0\,\prime} = \sum_{j=k+1}^{n} \sum_{l=1}^{r_{ij}} p_{ijl}^0 ,$$

we have

$$|\Delta_d^{\mathrm{R}}(h)| \leqslant C_1 C_2{}^R H \prod_{k=1}^{n-1} \int \cdots \int_{R_k} q_k^{0\,\prime - L_k} \prod_{j=k+1}^{n} \prod_{l=1}^{r_{kj}} \frac{dp_{kjl}}{2p_{kjl}^0}$$

$$= C_1 C_2{}^R H \prod_{k=1}^{n-1} I(R_k, L_k) \qquad \text{[cf. (2.3)]}$$

$$\leqslant C_1 \prod_{k=1}^{n-1} C_{2,k}^{R_k} m^{-L_k} \frac{R_k^{R_k - L_k} A_k^{L_k} L_k^{L_k \alpha}}{(R_k - 1)!\,(L_k - R_k \kappa)}$$

if $L_k = L(_k R_k) > R_k \kappa$.

For $\alpha \geqslant 1$ we choose $L_k(R_k) = \gamma R_k$, $\gamma > \kappa$ and obtain

$$|\Delta^{\mathrm{R}}{}_d(h)| \leqslant C_1 \prod_{k=1}^{n-1} \prod_{j=k+1}^{n} C_{2,k}^{r_{kj}} A_k^{r_{kj}\gamma} r_{kj}^{(\alpha-1)\gamma r_{kj}}$$

where (A1.7) was taken into account.

For $0 < \alpha < 1$ we choose $L_k(R_k) = R_k{}^\beta$, $1 < \beta < 1/\alpha$, and observing $(r_1 + r_2)^\beta \leqslant (2r_1)^\beta + (2r_2)^\beta$ we arrive at

$$|\Delta_d{}^{\mathrm{R}}(h)| \leqslant C_1 \prod_{k=1}^{n-1} \prod_{j=k+1}^{n} C_{2,k}^{r_{kj}} A_k^{r_{kj}{}^\beta} r_{kj}^{(\alpha\beta-1) r_{kj}{}^\beta}$$

Condition α of Theorem 1 is formulated by means of a positive sequence $m_\alpha(r)$, depending on α

$$m_\alpha(r) = \begin{cases} r^{-(\alpha-1)\gamma' r}, \ \kappa < \gamma', \text{ for } 1 < \alpha \\ \sigma(r)^r, \ \overline{\lim_{r\to\infty}}\, \sigma(r) = 0, \text{ for } \alpha = 1 \\ \exp(r^{\beta'}), \ 1 < \beta' < 1/\alpha, \text{ for } 0 < \alpha < 1 \end{cases}$$

In the estimations leading to Theorem 1 we have used the sequence

$$L_\alpha(r) = \begin{cases} \gamma r, \ \kappa < \gamma, \ 1 \leqslant \alpha \\ r^\beta, \ 1 < \beta < 1/\alpha, \ 0 < \alpha < 1 \end{cases}$$

Lemma 2

Assume the sequences $\{d_r^{(i)}\}$, $1 \leqslant i \leqslant n$, to satisfy condition α. Then

$$\left|\prod_{i=1}^{n} d_{S_i}^{(i)}\right| \leqslant \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} m_\alpha(r_{ij})\, r_{ij}!\, A_{ij}^{L_\alpha(r_{ij})}$$

for positive constants A_{ij}.

Proof: Simple estimates—the factorials are treated by means of Stirling's formula—show that

$$(r_1 + r_2)!\, m_\alpha(r_1 + r_2) \leqslant r_1!\, m_\alpha(r_1)\, A_1^{L_\alpha(r_1)}\, r_2!\, m_\alpha(r_2)\, A_2^{L_\alpha(r_2)}$$

Thus

$$|d_{S_i}^{(i)}| \leqslant \prod_{j=1}^{n} [r_{ij}!\, m_\alpha(r_{ij})\, A_{ij}^{L_\alpha(r_{ij})}]^{1/2}$$

In the product over i, each factor appears twice, therefore

$$\left|\prod_{i=1}^{n} d_{S_i}^{(i)}\right| \leqslant \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} r_{ij}!\, m_\alpha(r_{ij})\, A_{ij}^{r_{ij}}$$

The combination of Lemma 1 and Lemma 2 provides an estimate of the n-point function, which is the product of $\binom{n}{2}$ convergent series. Theorem A.1 shows that the convergence in the difference variables is equivalent to the convergence in the original variables. Thus we have arrived at the following statement.

Theorem 3

Assume the power series $\chi^{(i)}(g)$ satisfy condition α. Then all vacuum expectation values of these fields exist as functionals on $\mathscr{S}^\alpha$.

Since we have shown absolute convergence, we know that the multiple series corresponding to the n-point functions converge in each order of summation. Therefore, we may approximate the Wightman functions of the power series by those of Wick polynomials, which satisfy all axioms. In this way we could prove the Wightman axioms for the vacuum expectation values of the $\chi^{(i)}(g)$, and by the reconstruction theorem (Streater & Wightman, 1964) for the fields themselves. Instead of this procedure we prefer to construct the fields directly as operators in the Fock space and then to discuss their general properties.

4. *The Field Operators*

Let $P(\phi)$ denote a polynomial in the free Bose field (i.e., a multilinear functional, not a Wick polynomial, which is a linear functional). We define a dense domain in the Fock space for each $\alpha \in [0, \infty)$ as follows

$$D_0^\alpha := \{P(\phi)\Omega, \text{ test functions in } \mathscr{S}^\alpha\} \qquad (4.1)$$

If $\{d_r^{(i)}\}$ satisfies condition α, then the sequence of partial sums

$$\chi_N^{(i)}(g) = \sum_{r=0}^{N} (d_r^{(i)}/r!) :\phi^r:(g)$$

has a strong limit on $D_0{}^\alpha$: For each $F \in D_0{}^\alpha$ the norm distance squared $\|(\chi_{N_1}^{(i)}(g) - \chi_{N_2}^{(i)}(g))F\|^2$ consists of four terms which are in structure partial sums of (3.1), since the free field $\phi(g)$ itself satisfies condition α for each $\alpha \in [0, \infty)$. Theorem 3 shows that the limits N_1, $N_2 \to \infty$ exist. They cancel each other and $\chi_N^{(i)}(g)F$ is a Cauchy sequence and has, by means of the completeness of the Fock space, a strong limit.

In order to get an invariant domain let us consider

$$D_1{}^\alpha := \{P(\chi^{(i)})\Omega,\ \chi^{(i)} \text{ satisfying condition } \alpha, \text{ test functions in } \mathscr{S}^\alpha\} \tag{4.2}$$

Since $\phi(g)$ is a special $\chi^{(i)}(g)$, $D_1{}^\alpha \supset D_0{}^\alpha$ holds. By the same arguments as above one concludes that each $\chi(g)$ satisfying condition α exists as a strong limit on $D_1{}^\alpha$, and by definition

$$\chi(g)\, D_1{}^\alpha \subset D_1{}^\alpha$$

For the discussion of Lorentz invariance we introduce the usual representation of the inhomogeneous Lorentz group in the test function space, which reads, for example, in the momentum space representation

$$U(\Lambda, a)\, \tilde{g}(p) = \exp(iap)\, \tilde{g}(\Lambda^{-1} p)$$

The defining relation (A1.1) for $\tilde{g}(p)$ to be in $\mathscr{S}_{\alpha, A}(R^{\kappa+2})$ is not invariant under $U(\Lambda, a)$, since the constant A is affected by a Lorentz transformation. But in virtue of (A1.3) $\mathscr{S}_\alpha$ is invariant, and since $U(\Lambda, 0)$ commutes with the Fourier transformation, $\mathscr{S}^\alpha$ is invariant also. With $U(\Lambda, a)$ is associated a (highly reducible) representation of the inhomogeneous Lorentz group in the Fock space

$$\check{U}(\Lambda, a) := \sum_{n=0}^{\infty} U(\Lambda, a) \otimes \underset{n}{\cdots} \otimes U(\Lambda, a)$$

where the empty product equals the unit operator (invariant no-particle state). $D_1{}^\alpha$ is thus invariant against Lorentz transformations

$$\check{U}(\Lambda, a)\, D_1{}^\alpha = D_1{}^\alpha$$

Because of

$$[U^{-1}(\Lambda, a):\phi^r:](g) = :\phi^r:(U(\Lambda, a) g) = \check{U}(\Lambda, a):\phi^r:(g)\, \check{U}^{-1}(\Lambda, a)$$

we obtain the correct relativistic transformation law for the power series

$$[U^{-1}(\Lambda, a)\, \chi](g) = \check{U}(\Lambda, a)\, \chi(g)\, \check{U}^{-1}(\Lambda, a) \tag{4.3}$$

The spectral condition for the energy-momentum operator corresponding to $\check{U}(\Lambda, a)$ is trivially satisfied.

As for locality, let us consider the commutator

$$[\chi^{(1)}(g_1), \chi^{(2)}(g_2)]_- = \sum_{i_1, i_2=0}^{\infty} \sum_{r=0}^{\infty} \frac{d^{(1)}_{r+i_1} d^{(2)}_{r+i_2}}{r!\, i_1!\, i_2!} \times$$
$$\times \int\int [(i\Delta^{(+)}(x_1 - x_2))^r - (i\Delta^{(+)}(x_2 - x_1))^r] \times$$
$$\times :\phi^{i_1}(x_1)\, \phi^{i_2}(x_2): g_1(x_1)\, g_2(x_2)\, dx_1\, dx_2 \qquad (4.4)$$

[...] is an odd Lorentz invariant distribution and thus vanishes for space-like separations. Therefore, if the fields $\chi^{(i)}(g)$ are localisable (i.e., if they are defined on test functions of compact support) then they are local and relatively local to each other.

The asymptotic condition is discussed in accordance with Jost (1965). For this purpose we introduce the field

$$\chi_{as}(g) := \chi(g_{as})$$

where g_{as} is in momentum space equal to $\tilde{g}(p)h(p^2)$, and $h(s)$ is a test function such that in a neighbourhood of $s = m^2$, $h(s) = 1$, and $h(s) = 0$ for $|s - m^2| > m^2/2$. This auxiliary function h has the effect that all Wick powers of degree greater than one are annihilated. If we set in the series of $\chi(g)$ the constant d_0 equal to zero, then $\chi_{as}(g) = d_1 \phi(g)$ and the asymptotic condition as well as the asymptotic completeness are satisfied.

Since for each power series there is an $\alpha \in [0, \infty)$, so that condition α of Theorem 1 is satisfied, we have

Theorem 4

For each power series $\chi(g)$ there exists a functional domain $\mathscr{S}^{\alpha}$ and a dense, invariant operator domain $D_1{}^{\alpha}$ in the Fock space of the free field $\phi(g)$, where it is well-defined as an operator-valued functional and satisfies all Wightman axioms, with the possible exception of locality.

Fields which are functionals in $\mathscr{S}^{\alpha\prime}$, $1 < \alpha$, are localisable in the ordinary sense, because in the corresponding test function spaces there are functions of compact support. There are, moreover, several equivalent methods to define the support of a functional in $\mathscr{S}^{1\prime}$ (Martineau, 1963; Khoruzhij, 1966; Constantinescu, 1969), and we consider these fields as localisable, too. To the set of localisable power series belongs also the free field. Hence all $\chi(g)$ which are elements of $\mathscr{S}^{1\prime}$ are in the Borchers class (Borchers, 1960) of the free field. Recalling condition α for $\alpha = 1$ we have the following statement.

Theorem 5

All power series of order smaller than two and of order two and type zero are in the Borchers class of the free field. This holds for arbitrary space dimensions greater than one.

If $\chi(g)$ is of order two and of finite type, it belongs, seemingly, to a space $\mathscr{S}^{1,A'}$. (We have in Section 2 only shown that the two-point function lies in $\mathscr{S}^{1,A'}$.) These partially local fields are sometimes considered as belonging to theories with a built-in elementary length. In our description such a fundamental length would be connected with the constant A, which itself may depend on the coupling constant.

In order to exhaust the Borchers class of the free field one has only to include infinite series with Lorentz invariant derivatives into the considerations and to investigate which of them be localisable. That no other fields are to be taken into account is a consequence of Epstein's theorem (Epstein, 1963).

5. *Dynamical Properties*

In the foregoing section we were primarily concerned with those general properties of the power series $\chi(g)$, which allow us to call them fields in the sense of Wightman. The results of Section 2 and Section 3, however, enable us to determine the maximal functional domain for an arbitrary $\chi(g)$ within the accuracy, given by the family of test function spaces $\mathscr{S}^{\alpha}$. Such a programme was already discussed by Schroer (1964) in connection with infinite series of the zero-mass field, but without specifying the test function spaces. We now show how such considerations may lead to further statements on the fields, statements which belong to the dynamical part of a theory.

The expression (2.1) for the two-point function can be written in the form (assuming d_0 to be equal to zero)

$$\mathscr{W}^{(2)}(g_1, g_2) = \sum_{r=1}^{\infty} \frac{|d_r|^2}{r!} \int \sigma_r(q^2)\tilde{g}_1^*(q)\tilde{g}_2(q)\,dq \tag{5.1}$$

where the phase-space integral of r identical particles $\sigma_r(q^2)$ is given by

$$\sigma_r(q^2) = (2\pi)^{-(\kappa+1)(r-1)} \int \cdots \int \delta\left(q - \sum_{l=1}^{r} p_l\right) \prod_{l=1}^{r} \Theta(p_l^0)\,\delta(p_l^2 - m^2)\,dp_l$$

The $\sigma_r(q^2)$ are Lorentz invariant distributions which for $r > 1$ are equivalent to discontinuous, positive functions. Defining

$$\rho_N(q^2) = \sum_{r=2}^{N} \frac{|d_r|^2}{r!}\sigma_r(q^2)$$

we know $\rho_N(q^2) < \rho_{N+1}(q^2)$ and

$$\int \rho_N(q^2)\tilde{g}^*(q)\tilde{g}(q)\,dq < \mathscr{W}^{(2)}(g, g)$$

for all N. Thus we are able to apply the Lebesgue convergence theorem and to interchange summation and integration in (5.1), obtaining

$$\mathscr{W}^{(2)}(g, g) = \int \rho(q^2)\tilde{g}^*(q)\tilde{g}(q)\,dq \tag{5.2}$$

The series $\rho(q^2) = \sum_{r=1}^{\infty} (|d_r|^2/r!)\,\sigma_r(q^2)$ converges for those values of q, for which not all test functions of the space under consideration vanish. If the $\tilde{g}$ are in $\mathscr{D}$, $\rho(q^2)$ is defined for finite q's only, i.e., there is no function describing the behaviour of $\rho(q^2)$ for large values of q^2.

Let us assume $\chi(g)$ satisfies condition α. According to (A1.2) $\tilde{g}(q) \in \mathscr{S}_\alpha$ implies $|\tilde{g}(q)| \leqslant C\exp(-a|q_0|^{1/\alpha})$. Since the integral of (5.2) is finite, we know

$$\rho(q^2) < \exp[a(q^2)^{1/2\alpha}]$$

for all $a > 0$. Moreover, if the sequence of the series coefficients $\{d_r\}$ satisfies a condition of Theorem 2 for an α', $\alpha < \alpha'$, then (5.2) diverges for some test functions in $\mathscr{S}_{\alpha'}$. Thus

$$\exp[b(q^2)^{1/2\beta}] \leqslant \rho(q^2)$$

for all $b > 0$ and $\alpha' < \beta$. This provides the following theorem on infinite series of phase-space integrals for space dimensions greater than one ($\kappa > 0$).

Theorem 6

Assume the positive sequence $\{a_r\}$ to satisfy one of the following conditions

$$1/(\alpha'-1)\,\kappa \leqslant \rho\{a_r\} < 1/(\alpha-1)\,\kappa, \qquad 1 < \alpha < \alpha' < \beta$$

$$\rho\{a_r\} = \infty, \qquad (1 = \alpha < \beta)$$

$$\exp(r^{1/\alpha'}) \leqslant a_r < \exp(r^{1/\alpha''}), \qquad 0 < \alpha < \alpha'' < \alpha' < \beta \leqslant 1, \qquad r > r_0$$

then the infinite series of phase-space integrals is estimated for large values of q^2 by

$$\exp[b(q^2)^{1/2\beta}] \leqslant \sum_{r=1}^{\infty} a_r\,\sigma_r(q^2) < \exp[a(q^2)^{1/2\alpha}] \tag{5.3}$$

for arbitrary $a, b > 0$.

In order to get the usual spectral representation of the two-point function, one has to observe that $\rho(q^2)$ is localised on the forward-cone. Such distributions can be written (Rieckers & Güttinger, 1968) as

$$\mathscr{W}^{(2)}(g_1, g_2) = \int_0^\infty \rho(\mu^2) \int \theta(q^0)\,\delta(\mu^2 - q^2)\,\tilde{g}_1{}^*(q)\,\tilde{g}_2(q)\,dq\,d\mu^2$$

or in position space

$$\mathscr{W}^{(2)}(g_1, g_2) = \int_0^\infty \rho(\mu^2) \int\int g_1{}^*(x)\,i\Delta^{(+)}(x-y, \mu^2)\,g_2(y)\,dx\,dy\,d\mu^2$$

These spectral representations are well defined, there is no need for a weighting function to compensate the growth of $\rho(\mu^2)$, $\mu^2 \to +\infty$. From the above discussion of the two-point function one gets useful information for the time-ordered two-point function

$$T^{(2)}(g_1, g_2) = \sum_{r=1}^{\infty} \frac{|d_r|^2}{r!} \int \int g_1^*(x) \Delta_F{}^r(x-y) g_2(y)\, dx\, dy$$

$$= \int_0^{\infty} \rho'(\mu^2) \int \int g_1^*(x) \Delta_F(x-y, \mu^2) g_2(y)\, dx\, dy\, d\mu^2$$

which is more difficult to treat. The spectral function $\rho'(\mu^2)$ is a generalised function of an unusual kind, but related to $\rho(\mu^2)$ in a specific manner. The detailed investigation of this problem is deferred to another occasion.

In contrast to the growth of the spectral functions, the singular behaviour at the light-cone can only be studied for localisable power series. As is shown in Theorem A2, the local vacuum expectation values are boundary values of analytic functions. Approaching the boundary of holomorphy the analytic functions tend to infinity if the real part of the complex vector lies on the light-cone. The degree of singularity which may occur is determined by the test function space.

Theorem 7

Let the power series $\chi^{(i)}(g)$, $i = 1, \ldots, n$, satisfy condition α, $\alpha \geqslant 1$. Then the corresponding vacuum expectation values transformed to difference variables

$$W^{(n-1)}(h^{(n-1)}) = (U_{21}\, U_{10}\, \mathscr{W}^{(n)})(h^{(n-1)}) = \mathscr{W}^{(n)}(g_1, \ldots, g^n)$$

are boundary values of functions

$$W^{(n-1)}(\xi - i\eta) = W^{(n-1)}(\xi_{1-i}\, \eta_1, \ldots, \xi_{n-1} - i\eta_{n-1})$$

which are analytic in $\mathbf{R}^{(\kappa+2)(n-1)} - i\Gamma$, Γ being the direct product of the forward-cones, i.e.,

$$W^{(n-1)}(h^{(n-1)}) = \lim_{\substack{\eta \to 0 \\ \eta \in \Gamma}} \int W^{(n-1)}(\xi - i\eta)\, h^{(n-1)}(\xi)\, d\xi$$

For each compact set $K \subset \Gamma$ there is a polynomial $P_K(\xi)$ and a constant $C > 0$, so that

$$|W^{(n-1)}(\xi - i\eta t)| \leqslant P_K(\xi) \exp\,[Ct^{1/(1-\alpha)}], \qquad 0 < t < 1 \qquad (5.4)$$

For $\alpha = 1$ this means that the singular behaviour may be arbitrarily strong.

Having the explicit expression (3.1) for $\mathscr{W}^{(n)}$ we are able to write down the series expansion of $W^{(n-1)}(\xi - i\eta)$, too. Since $W^{(n-1)}(\xi - i\eta)$ is the Laplace transform of the $\tilde{W}^{(n-1)}$, and the Laplace transformation commutes for generalised functions with infinite summation, we know

$$\int W^{(n-1)}(\xi - i\eta)\, h^{(n-1)}(\xi)\, d\xi = \sum_{\mathbf{R}\geqslant 0} \frac{d_{S_1}^{(1)}\cdots d_{S_n}^{(n)}}{\mathbf{R}!} \int \mathscr{L}(\tilde{\Delta}_d^{\mathbf{R}}) \times$$
$$\times (\xi - i\eta)\, h^{(n-1)}(\xi)\, d\xi$$

where $\tilde{\Delta}_d^{\mathbf{R}}$ is the Fourier transform of $\Delta_d^{\mathbf{R}}$ [cf. (3.1)]. If we denote

$$i\Delta^{(+)}(\xi_k - i\eta_k) = (2\pi)^{-(\kappa+1)} \int \exp[-i(\xi_k - i\eta_k)p]\,\Theta(p_0)\,\delta(p^2 - m^2)\, dp$$

then

$$\mathscr{L}(\tilde{\Delta}_d^{\mathbf{R}})(\xi - i\eta) = \Delta_d^{\mathbf{R}}(\xi - i\eta)$$
$$= \prod_{1\leqslant i<j\leqslant n} [i\Delta^{(+)}(\xi_i + \cdots + \xi_{j-1} - i(\eta_i + \cdots + \eta_{j-1}))]^{r_{ij}}$$

Since, according to (A1.9), the Laplace transforms differ from the Fourier transforms only by a modification of the test function, absolute convergence for the former implies that for the latter, and we may again apply the Lebesgue theorem to obtain

$$\int W^{(n-1)}(\xi - i\eta)\, h^{(n-1)}(\xi)\, d\xi = \int \left[\sum_{\mathbf{R}\geqslant 0} \frac{d_{S_1}^{(1)}\cdots d_{S_n}^{(n)}}{\mathbf{R}!} \Delta_d^{\mathbf{R}}(\xi - i\eta)\right] h^{(n-1)}(\xi)\, d\xi$$

from which follows the equality of the factors beside $h^{(n-1)}$. Thus we may state:

Theorem 8

If the sequences $\{d_r^{(i)}\}$, $i = 1, \ldots, n$, satisfy condition α, $\alpha \geqslant 1$, then the multiple series

$$\sum_{\mathbf{R}\geqslant 0} \frac{d_{S_1}^{(i)}\cdots d_{S_n}^{(n)}}{\mathbf{R}!} \prod_{1\leqslant i<j\leqslant n} [i\Delta^{(+)}(\xi_i + \cdots + \xi_{j-1} - i(\eta_i + \cdots + \eta_{j-1}))]^{r_{ij}}$$

converges absolutely in $\mathbf{R}^{(\kappa+2)n} - i\Gamma$ to a function, which is analytic in this region and satisfies the inequality (5.4).

Remark: The convergence in Theorem 8 can be shown to be uniform in each compact subset of $\mathbf{R}^{(\kappa+2)n} - i\Gamma$.

By means of Theorem 8 the convergence of functionals is transformed to the convergence of analytic functions. The correspondence between these two kinds of convergence was studied by Jaffe in more general terms (Jaffe, 1965b, 1968) and forms the basis for his investigation of power series in two dimensions. The advantage of our method is that it works for nonlocal fields also.

6. *Operator-Analysis*

We have now at hand a set of power series, defined as operator-valued functionals. This set is even larger than the set of convergent numerical power series, since in the functional case for each growth of the coefficients suitable domains of definition can be set up. It is not difficult to show that there hold analogous formulae as in the numerical analysis.

If, for example, $\{d_r\}$ satisfies condition α, then $\{rd_r\}$ does also (for the same α), and the differentiation rule for Wick powers leads to

$$\frac{\partial}{\partial x^\mu}\chi(g) = \sum_{r=1}^{\infty} \frac{d_r}{(r-1)!} :\phi^{r-1} \frac{\partial}{\partial x^\mu}\phi:(g) \tag{6.1}$$

Also, the multiplication law for power series

$$:\chi^{(1)}\chi^{(2)}:(g) = \sum_{r=0}^{\infty} \frac{1}{r!}\left(\sum_{s=0}^{r}\binom{r}{s} d_s^{(1)} d_{r-s}^{(2)}\right):\phi^r:(g) \tag{6.2}$$

is easily verified on those domains, where the right-hand side converges. Thus, we have a unique product at the same point of two field-operators, which possess essential, or even nonlocalisable, singularities.

This operator-analysis can be applied to field-theoretic models, where the solutions of the field equations are given by transcendental functions of the free field. For the derivative-coupling between a Fermi field ψ and a Bose field ϕ

$$\mathscr{L}_{\text{int}} = \lambda :\bar{\psi}\gamma^\mu \psi \frac{\partial}{\partial x^\mu} F(\phi):$$

the solution reads

$$\chi = :\exp[i\lambda F(\phi)]\psi:$$

χ is local only if $F(t)$ is entire and satisfies $|F| < ct^2$. Choosing various functions F all types of local and nonlocal fields are created. For $F(t) = t^2$ one obtains a partially local field, which has as appropriate functional domain a space $\mathscr{S}^{1,A}$, A depending on the coupling constant λ (Rieckers, 1969).

Transcendental functions of the free field occur also in those interactions which arise from chiral-invariance arguments. Examples are

$$\mathscr{L}_{\text{int}} = \bar{\psi} U(\boldsymbol{\varphi})\psi$$

where U may be one of the following functions (Gürsey, 1968)

1. $U = \exp(2if\gamma_5 \boldsymbol{\tau\varphi})$
2. $U = \dfrac{1 + if\gamma_5 \boldsymbol{\tau\varphi}}{1 - if\gamma_5 \boldsymbol{\tau\varphi}}$
3. $U = \sqrt{(1 - 4f^2 \boldsymbol{\varphi}^2)} + 2if\gamma_5 \boldsymbol{\tau\varphi}$

$\boldsymbol{\varphi}$ denoting the pion iso-triplet.

At the present time much effort is directed towards developing new mathematical methods for dealing with these nonpolynomial interactions (Salam, 1969a; Efimov, 1969a, b; Lee & Zumino, 1969; Salam & Strathedee, 1969). Generally, the time-ordered vacuum expectation values are attacked directly, but these quantities are uniquely defined only for those test functions which vanish sufficiently often if some coordinates coincide. In this context regularisation is the construction of an extension to the whole test function space. The functional-theoretic background of the extension procedure is fully clarified in the case of algebraic singularities, i.e., for renormalisable theories or for nonrenormalisable theories up to a finite order of perturbation theory. Convergence theorems for infinite series of Feynman amplitudes with increasing singularities—as they appear in non-polynomial Lagrangian theories already in the first order of perturbation theory—are not proven hitherto. Our investigations supply some information for this problem. Let us recall that we have shown an infinite power series $\chi(g)$ to converge in momentum space only for those test functions which decrease in the energy variable like $\exp(-a|p_0|^{1/\alpha})$. There is no possibility to define χ at the point x, which would correspond to the limit $\tilde{g}(p) \to \exp(ipx)$, i.e., to an oscillating function. In the same way, one can expect the time-ordered, regularised vacuum expectation values of the power series to converge only if they are smeared with appropriate test functions. Now, time-ordered products coincide—at least for localisable fields—in certain space-time regions with the non-ordered products. Thus, they can be smeared with test functions only, which satisfy the same (or stronger) conditions as in the non-ordered case, the actual convergence still remaining to be shown.

Appendix 1

Definition and Properties of the Spaces of Type S

According to Gelfand & Schilow (1962) an infinitely differentiable function $\tilde{g}(p)$, $p=(p_1,\ldots,p_n)$, is said to be in the space $\mathscr{S}_{\alpha,A}(R^n)$, where $\alpha=(\alpha_1,\ldots,\alpha_n)$ and $A=(A_1,\ldots,A_n)$, if and only if

$$\begin{aligned}|p^k D^q \tilde{g}(p)| := &\left| p_1^{k_1}\cdots p_n^{k_n} \frac{\partial^{q_1+\cdots+q_n}}{\partial p_1^{q_1}\cdots\partial p_n^{q_n}} \tilde{g}(p)\right| \\ &\leqslant C_{q_1\cdots q_n,\delta_1\cdots\delta_n} \times (A_1+\delta_1)^{k_1}\cdots(A_n+\delta_n)^{k_n} k_1^{k_1\alpha_1}\cdots k_n^{k_n\alpha_n} \\ &= : C_{q,\delta}(A+\delta)^k \end{aligned} \qquad \text{(A1.1)}$$

is valid for all k and q and arbitrary $\delta > 0$, the constant $C_{q,\delta}$ depending on $\tilde{g}$.† (A1.1) is equivalent to

$$|D^q\tilde{g}(p)| \leqslant B_{q,\delta} \prod_{i=1}^{n} \exp(-a_i(\delta_i)\,|p_i|^{1/\alpha_i}) \qquad a_i(\delta_i) = \alpha_i/e(A_i+\delta_i)^{1/\alpha_i} \qquad \text{(A1.2)}$$

This implies the existence of the countable set of norms

$$\|\tilde{g}\|_s^{\alpha,A} := \sup_{q\leqslant s} \prod_{i=1}^{n} \exp[a_i(1-1/s_i)\,|p_i|^{1/\alpha_i}]\,|D^q\tilde{g}(p)|$$

where $a_i = a_i(0)$. By means of this system of norms one introduces the topology for countable normed vector spaces in the standard way. A

† The spaces $\mathscr{S}_{\alpha,A}$ are used for the momentum space representation.

sequence $\tilde{g}_n \in \mathscr{S}_{\alpha, A}$ converges to zero in this topology if and only if all derivatives $D^q \tilde{g}_n(p)$ tend to zero uniformly in every finite interval and if the sequence of norms $\|\tilde{g}_n\|_s^{\alpha, A}$ is bounded for each fixed s.

For $A_1 < A_2$ holds $\mathscr{S}_{\alpha, A_1} \subset \mathscr{S}_{\alpha, A_2}$, and one denotes the inductive limit by

$$\mathscr{S}_\alpha := \bigcup_A \mathscr{S}_{\alpha, A} \tag{A1.3}$$

A sequence $\tilde{g}_n \in \mathscr{S}_\alpha$ converges to zero if and only if all $\tilde{g}_n$ are elements of a fixed space $\mathscr{S}_{\alpha, A}$ and tend to zero in the topology of $\mathscr{S}_{\alpha. A}$.

The set of all linear, continuous functionals on $\mathscr{S}_{\alpha, A}$ resp. $\mathscr{S}_\alpha$ is denoted by $\mathscr{S}'_{\alpha, A}$ resp. $\mathscr{S}_\alpha'$. An important property of these dual spaces is their completeness: Let T_n be a sequence of functionals in $\mathscr{S}'_{\alpha, A}$ resp. $\mathscr{S}_\alpha'$. If we show convergence for all numerical sequences $T_n(g)$, g varying in the corresponding test function space, then the limits $T(g)$ define a linear, continuous functional.

Via Fourier transformation

$$g(x) = (\mathscr{F}^{-1}\tilde{g})(x) = (2\pi)^{-n/2} \int \exp(-ipx)\tilde{g}(p)\, d^n p$$

we obtain the spaces

$$\mathscr{S}^{\alpha, A} = \mathscr{F}^{-1} \mathscr{S}_{\alpha, A}, \qquad \mathscr{S}^\alpha = \mathscr{F}^{-1} \mathscr{S}_\alpha \tag{A1.4}$$

An infinitely differentiable function $g(x)$ belongs to $\mathscr{S}^{\alpha, A}$ if and only if

$$|x^k D^q g(x)| \leqslant C_{k, \delta}(A + \delta)^q q^{q\alpha} \tag{A1.5}$$

For $0 < \alpha_1 < \alpha_2 < \infty$ we have the following relations

$$\mathscr{D} = \mathscr{S}_0 \subset \mathscr{S}_{\alpha_1} \subset \mathscr{S}_{\alpha_2} \subset \mathscr{S}$$
$$\mathscr{D}' = \mathscr{S}_0' \supset \mathscr{S}'_{\alpha_1} \supset \mathscr{S}'_{\alpha_2} \supset \mathscr{S}$$

and

$$\mathscr{Z} = \mathscr{S}^0 \subset \mathscr{S}^{\alpha_1} \subset \mathscr{S}^{\alpha_2} \subset \mathscr{S}$$
$$\mathscr{Z}' = \mathscr{S}^{0\prime} \supset \mathscr{S}^{\alpha_1\prime} \supset \mathscr{S}^{\alpha_2\prime} \supset \mathscr{S}$$

It should be noted that the union $\bigcup_\alpha \mathscr{S}_\alpha$ is smaller than $\mathscr{S}$. $\mathscr{S}^\alpha$ contains test functions of compact support if and only if $\alpha > 1$.

Positive definiteness in $\mathscr{S}^{\alpha\prime}$ and positivity in $\mathscr{S}_\alpha'$ are defined and connected by a generalised Bochner-Schwartz theorem in the same way as in $\mathscr{S}'$, and provide no additional difficulties (Gelfand & Wilenkin, 1964). It is therefore somewhat misleading to say that the positive-definiteness condition in the Wightman axioms excludes essential singularities.

We investigate now the problem of how to pass to difference variables in the case of a translation invariant functional in $\mathscr{S}^{\alpha\prime}$, and for this purpose introduce the following mappings in the test function spaces.

$$(U_{01} g)(x_1, \ldots, x_n) := g(x_1 - x_2, \ldots, x_{n-1} - x_n, x_n)$$
$$(U_{10} g)(\xi_1, \ldots, \xi_n) := g\left(\sum_{i=1}^n \xi_i, \sum_{i=2}^n \xi_i, \ldots, \xi_n\right) \tag{A1.6}$$

Evidently, $U_{01} = U_{10}^{-1}$.

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Lemma A1

U_{10} and U_{01} map the spaces $\mathscr{S}^{\alpha}$, $\alpha = (\alpha, \ldots, \alpha)$ into themselves.

Proof: We know that U_{10} and U_{01} map $\mathscr{S}$ into $\mathscr{S}$. Thus we have only to show that the estimate (A1.5) for g implies the same relation for $U_{10}g$ and $U_{10}g$, the constant A possibly having another value. By successive differentiation we obtain

$$|\xi^k D^q(U_{10}g)(\xi)| = \Bigg|\xi^k \sum_{\nu_1{}^1=q_1} \cdots \sum_{\nu_1{}^n+\cdots+\nu_n{}^n=q_n} \frac{q_1!}{\nu_1{}^1!} \cdots \frac{q_n!}{\nu_1{}^n! \cdots \nu_n{}^n!} \times \times \frac{\partial^{\mu_1+\cdots+\mu_n}}{\partial x^{\mu_1} \cdots \partial x^{\mu_n}} g(x_1, \ldots, x_n)\Bigg|_{x_j=\sum_{i=j}^{n}\xi_i}$$

where $\mu_j = \sum_{i=j}^{n} \nu_j{}^i$ and $\sum_{j=1}^{n} \mu_j = \sum_{i=1}^{n} q_i$. We now apply (A1.5) to g, and observe that

$$\mu_1^{\mu_1\alpha} \cdots \mu_n^{\mu_n\alpha} \leqslant \left(\sum_{i=1}^{n} q_i\right)^{\left(\sum_{i=1}^{n} q_i\right)\alpha} \leqslant \prod_{i=1}^{n} \exp\left[(n+1-i)q_i\alpha\right] q_i^{q_i\alpha} \quad \text{(A1.7)}$$

for arbitrary $\alpha \geqslant 0$, if the numbers q_i are sufficiently large. That gives the desired inequality

$$|\xi^k D^q(U_{10}g)(\xi)| \leqslant C_k \bar{A}^q q^{q\alpha}$$

where $\bar{A}_l = l(\prod_{j=1}^{l} A_l)\exp[(n+1-l)\alpha]$, the A_l denoting the constants of (A1.5) which correspond to $g(x)$. [Since we are working in $\mathscr{S}^{\alpha} = \bigcup_{A} \mathscr{S}^{\alpha, A}$, we have dropped the δ, which appears in (A1.5)]. An analogous estimation holds for $U_{01}g$.

Remark: The mappings U_{10} and U_{01} do not leave the spaces $\mathscr{S}^{\alpha, A}$ invariant.

*Theorem A*1

There is a one-to-one, continuous mapping which associates a functional $T_d \in \mathscr{S}^{\alpha\prime}(\mathbf{R}^{n-1})$ with each translation invariant functional $T \in \mathscr{S}^{\alpha\prime}(\mathbf{R}^n)$. [This is a modification of a statement of Streater & Wightman (1964) for translation invariant functionals in $\mathscr{S}'$.]

Proof: Let us denote by $Q_a{}^k$ a translation in the kth variable about a, and by Q_a a translation in all variables simultaneously. Suppose $T \in \mathscr{S}^{\alpha\prime}(\mathbf{R}^n)$ and translation invariant in all variables. Then

$$\begin{aligned}(Q_a{}^n U_{10} T)(g) &= T(U_{01} Q_{-a}^n g) = T(Q_{-a} U_{01} g) \\ &= (Q_a T)(U_{01} g) = (U_{10} T)(g)\end{aligned}$$

i.e. $U_{10}T$ is constant in the nth variable, and thus $\partial/\partial x_n(U_{10}T) = 0$. A further mapping which reduces the number of variables is defined by

$$(U_{21} T)(h) := T(h \otimes e)$$

where $h \in \mathscr{S}^{\alpha}(\mathbf{R}^{n-1})$ and $e \in \mathscr{S}^{\alpha}(\mathbf{R}^1)$, so that $\int e(\xi_n)\, d\xi_n = 1$. $U_{21}T$ is in $\mathscr{S}^{\alpha\prime}(\mathbf{R}^{n-1})$ if $T \in \mathscr{S}^{\alpha\prime}(\mathbf{R}^n)$. For a $T_2 \in \mathscr{S}^{\alpha\prime}(\mathbf{R}^{n-1})$ we introduce

$$(U_{12}T_2)(g) := T_2(\xi_1, \ldots, \xi_{n-1})\left(\int g(\xi_1, \ldots, \xi_n)\, d\xi_n\right)$$

and remark that $U_{12}T_2 \in \mathscr{S}^{\alpha\prime}(\mathbf{R}^n)$. Let us then calculate

$$(U_{12}U_{21}T_1 - T_1)(g) = T_1\left(\int g(\xi_1, \ldots, \xi_n)\, d\xi_n\, e(\xi_n) - g(\xi_1, \ldots, \xi_n)\right)$$
$$= : T_1(\bar{g}(\xi_1, \ldots, \xi_n))$$

Because of $\int \bar{g}(\xi_1, \ldots, \xi_n)\, d\xi_n = 0$ we have

$$f(\xi_1, \ldots, \xi_n) = \int_{-\infty}^{\xi_n} g(\xi_1, \ldots, \xi_n')\, d\xi_n'$$

is in $\mathscr{S}$. Since

$$\left|\frac{\partial q^n}{\partial \xi_n^{q_n}} f\right| = \left|\frac{\partial^{q_n - 1}}{\partial \xi_n^{q_n - 1}} \bar{g}\right| \leqslant C A^{q_n - 1}(q_n - 1)^{(q_n - 1)} \leqslant C' A^{q_n} q_n^{q_n \alpha}, \qquad q_n \geqslant 2$$

f is in $\mathscr{S}^{\alpha}(\mathbf{R}^n)$. We now set T_1 equal to $U_{10}T$ and obtain

$$(U_{12}U_{21}T_1 - T_1)(g) = T_1\left(\frac{\partial}{\partial \xi_n} f\right) = 0$$

since T_1 is constant in ξ_n if T is translation invariant. Thus we have shown that the mapping

$$T_d = U_{21}U_{10}T$$

associates a functional $T_d \in \mathscr{S}^{\alpha\prime}(\mathbf{R}^{n-1})$ with each translation invariant functional $T \in \mathscr{S}^{\alpha\prime}(\mathbf{R}^n)$ and has a unique inverse. The continuity of the mapping and its inverse results from general properties of mappings in dual spaces (cf., e.g., Schaefer, 1966).

For functionals in $\mathscr{S}^{\alpha\prime}$, $\alpha \geqslant 1$, the spectral condition implies a representation as boundary values of analytic functions, in analogy to the case of the tempered distributions (Streater & Wightman, 1964). In order to show this we introduce, besides the Fourier transformation for functionals,

$$T(g) = (\mathscr{F}^{-1}\tilde{T})(g) = \tilde{T}(\mathscr{F}g) = \tilde{T}(\tilde{g}) \tag{A1.8}$$

the Laplace transformation also

$$(\mathscr{L}_\eta \tilde{T})(g) := \{\mathscr{F}^{-1}[\exp(-p\eta)\tilde{T}]\}(g) = \tilde{T}[\exp(-p\eta)\tilde{g}], \qquad p, \eta \in \mathbf{R}^{(\kappa+2)n} \tag{A1.9}$$

Let V^+ be the forward cone in $\mathbf{R}^{(\kappa+2)n}$. We denote the direct product by

$$\Gamma := \prod_{i=1}^{n} \otimes V_i^+ \tag{A1.10}$$

Theorem A2†

Let $\tilde{T} \in \mathscr{S}_\alpha'(\mathbf{R}^{(\kappa+2)n})$, $\alpha = (\alpha, \ldots, \alpha)$ with $\alpha \geqslant 1$, and supp$\tilde{T} \subset \Gamma$. Then the Laplace transform $\mathscr{L}_\eta(T)$ exists for all $\eta \in \Gamma$ and is equivalent to an holomorphic function $\mathscr{L}(\tilde{T})(\xi - i\eta)$. For $\eta \to 0$ within Γ, $\mathscr{L}(\tilde{T})(\xi - i\eta)$ converges to $T = \mathscr{F}^{-1}\tilde{T}$ in the topology of $\mathscr{S}^{\alpha'}$. Let K be a compact subset of Γ. Then there exists a polynomial P_K and a positive constant C, so that

$$|\mathscr{L}(\tilde{T})(\xi - i\eta t)| \leqslant P_K(\xi) \exp[Ct^{1/(1-\alpha)}] \tag{A1.11}$$

holds for all $\xi \in \mathbf{R}^{(\kappa+2)n}$, all $\eta \in K$, and for $t \in (0,1)$. For $\alpha = 1$ that means that the singularity of the left-hand side, which arises if t goes to zero, may be arbitrarily strong.

Proof: (i) Let $\gamma(p)$ be an infinitely differentiable auxiliary function, which is equal to one in a region containing Γ and which vanishes outside a larger region of bounded distance to Γ. Hence $\tilde{T}(\tilde{g}) = \tilde{T}(\gamma\tilde{g})$. The function $\exp(-p\eta)\gamma(p)$, $\eta \in \Gamma$, belongs to $\mathscr{S}_{1, A(\eta)}$ as well as the sequence

$$\tilde{h}_\nu(p) = \exp(-p\eta)\gamma(p)\tilde{g}_\nu(p)$$

$\tilde{g}_\nu \in \mathscr{S}$. If $\tilde{g}_\nu \to 0$ in $\mathscr{S}$ then $\tilde{h}_\nu \to 0$ in $\mathscr{S}_1$ and at the same time in all $\mathscr{S}_\alpha$, $\alpha \geqslant 1$. Thus we have shown that $\exp(-p\eta)\tilde{T}(\tilde{g}) = \tilde{T}[\exp(-p\eta)\gamma\tilde{g}]$ exists for all $\tilde{g} \in \mathscr{S}$ and is continuous, i.e. $\exp(-p\eta)\tilde{T} \in \mathscr{S}'$.

(ii) We are now able to apply Theorem 2-6 of Streater & Wightman (1964) and deduce that $\mathscr{L}_\eta(\tilde{T})$ exists and is equivalent to a function which is holomorphic in $\mathbf{R}^{(\kappa+2)n} - i\Gamma$ and satisfies

$$|\mathscr{L}(\tilde{T})(\xi - i\eta)| \leqslant P_K(\xi), \qquad \eta \in K$$

For $\eta \to 0$ in Γ, $D^q\{[\exp(-p\eta) - 1]\gamma(p)\}$ tends to zero uniformly in p, and if $\tilde{g} \in \mathscr{S}_\alpha$, then $\tilde{g}(p)[\exp(-p\eta) - 1]\gamma(p)$ tends to zero in the topology of $\mathscr{S}_\alpha$. Thus we have by the continuity of $\tilde{T}$

$$\lim_{\eta\to 0} \int \mathscr{L}(\tilde{T})(\xi - i\eta)g(\xi)\,d\xi = \lim_{\eta\to 0} \tilde{T}[\exp(-p\eta)\tilde{g}(p)\gamma(p)] = \tilde{T}(\tilde{g}) = T(g)$$

(iii) We now make use of the function

$$a(p, \eta, \eta_j) = \exp(-p\eta)\left[\sum_{j=1}^{l} \exp(-p\eta_j)\right]^{-1}$$

introduced by Streater & Wightman (1964). η_j are to be chosen in Γ in the way that the convex hull $H = \{\eta', \eta' = \sum_{j=1}^{l} t_j\eta_j, \sum_{j=1}^{l} t_j = 1\}$ has a non-empty interior. As is shown by Jaffe (1968), $a(p, \eta, \eta_j)$ is infinitely often differentiable in p and η and satisfies

$$|D_p^m a(p, \eta, \eta_j)| \leqslant C_m \exp(-d\|p\|)$$

for all $\eta \in H$, where $d > 0$ and $\|p\|^2 = \sum_{i=1}^{(\kappa+2)n} p_i^2$.

† Theorem A.2 is intimately connected with a more general theorem on Jaffe (1968) for strictly localisable functionals. We have adapted the statement for functionals in $\mathscr{S}^{\alpha'}$ and shortened the proof considerably by requiring the stronger but physically motivated assumption supp$\tilde{T} \subset \Gamma$.

Because $\mathscr{L}_\eta(\tilde{T})$ is the Fourier transform of a rapidly decreasing distribution, there holds for the corresponding analytic function

$$\mathscr{L}(\tilde{T})(\xi - i\eta) = \tilde{T}(\exp(-ip\xi - pt\eta))$$

and

$$\mathscr{L}(\tilde{T})(\xi - i\eta) = \tilde{T}_1{}^t(\exp(-ip\xi)\,a(p, t\eta, t\eta_j))$$

where

$$\tilde{T}_1{}^t = \left[\sum_{j=1}^{l} \exp(-pt\eta_j)\right]\tilde{T}$$

$\{\tilde{T}_1{}^t, 0 < t < 1\}$ is a bounded set of functionals in $\mathscr{S}_\alpha{}'$ (Gelfand & Wilenkin, 1964) and this implies the existence of positive constants B, A and s, so that

$$|\tilde{T}_1{}^t(\exp(-ip\xi)\,a(p, t\eta, t\eta_j))| \leqslant B\|\exp(-ip\xi)\,a(p, t\eta, t\eta_j)\|_s^{\alpha, A}$$

Without restricting in generality we suppose $A = (A, \ldots, A)$ and $s = (s, \ldots, s)$. Observing

$$\sum_{i=1}^{(\kappa+2)n} |p_i|^{1/\alpha} \leqslant 2^{(\kappa+2)n}\|p\|^{1/\alpha}$$

and

$$|D^q[\exp(-ip\xi)\,a(p, t\eta, t\eta_j)]| = |D^q[\exp(-ip\xi)\,a(tp, \eta, \eta_j)]| \leqslant P_{q,\eta}(\xi)\exp(-td\|p\|)$$

we have

$$\begin{aligned}\|\exp(-ip\xi)\,a(p, t\eta, t\eta_j)\|_s^{\alpha, A} &= \sup_{q\leqslant s}\exp\left[a(1 - 1/s)\sum_{i=1}^{(\kappa+2)n} |p_i|^{1/\alpha}\right]\times \\ &\quad \times |D^q[\exp(-ip\xi)\,a(p, t\eta, t\eta_j)]| \\ &\leqslant P_{s,\eta}(\xi)\exp[b\|p\|^{1/\alpha} - dt\|p\|] \\ &\leqslant P_{s,\eta}(\xi)\exp[Ct^{1/(1-\alpha)}], \text{ for } \alpha > 1\end{aligned}$$

Since η is varying in H, there is a maximal polynomial depending on H. Each compact subset K of Γ, however, can be enclosed in an $H \subset \Gamma$. Thus we have finally

$$|\mathscr{L}(\tilde{T})(\xi - it\eta)| \leqslant P_K(\xi)\exp[Ct^{1/(1-\alpha)}]$$

By means of Theorem A2, most results of the general quantum field theory (Streater & Wightman, 1964; Jost, 1965) can be obtained also for fields in $\mathscr{S}^{\alpha\prime}$, $\alpha > 1$. The case $\alpha = 1$ requires some modifications, but localisability can be defined in various ways (Martineau, 1963; Khoruzhij, 1966; Constantinescu, 1969). Equation (A1.11) characterises the singular behaviour of the vacuum expectation values at the light-cone.

Appendix 2

Power Series in Two Dimensions

For $\kappa = 0$ the integral (2.3) reads

$$I(r,L) = \int_0^\infty \cdots \int_0^\infty \left(\sum_{l=1}^r p_l^0\right)^{-L} \prod_{l=1}^r \frac{dp_l}{p_l^0}, \qquad p_l^0 = (p_l^2 + m^2)^{1/2}$$

The rules for the existence of multiple integrals (cf., e.g., Weinberg, 1960) show that $I(r,L)$ is finite for each $L > 0$. Thus L must not necessarily depend on r. Because of $(p_l^2 + m^2)^{1/2} \geqslant 2^{-1/2}(p_l + m)$

$$I(r,L) \leqslant 2^{(L+r)/2} \int_0^\infty \cdots \int_0^\infty \left(\sum_{l=1}^r (p_l + m)\right)^{-L} \prod_{l=1}^r \frac{dp_l}{p_l + m}$$

$$= 2^{(L+r)/2} \int_{rm}^\infty \cdots \int_{2m}^{t_3-m} \int_m^{t_2-m} t_r^{-L} \prod_{l=2}^r \frac{dt_l}{t_l - t_{l-1}} \frac{dt_1}{t_1}$$

where we have set $t_k = \sum_{l=1}^k (p_l + m)$, $1 \leqslant k \leqslant r$. Taking into account

$$\int_m^{t_2-m} \frac{dt_1}{(t_2 - t_1)\, t_1} \leqslant \frac{1}{m} \log \frac{t_r}{m}$$

$$\int_{lm}^{t_{l+1}-m} \frac{dt_l}{t_{l+1} - t_l} \leqslant \log \frac{tr}{m}, \qquad 2 \leqslant l \leqslant r-1$$

we obtain

$$I(r,L) \leqslant 2^{(L+r)/2} m^{-1} \int_{rm}^\infty t_r^{-L} \log^{(r-1)} \frac{t_r}{m}\, dt_r$$

$$= 2^{(L+r)/2} m^{-L} \frac{(r-1)!}{(L-1)^r} r^{-L+1} \sum_{\rho=0}^{r-1} \frac{(L-1)^{r-1-\rho}}{(r-1-\rho)!} \log^{r-1-\rho} r, \qquad L > 1$$

Replacing the sum by r times the maximal member we have

$$I(r,L) \leqslant C 2^{r/2} m^{-L} (L-1)^{-r}\, r!$$

for large r. Supposing

$$|p_0{}^L \tilde{g}_1(p) \tilde{g}_2(p)| \leqslant H(L)$$

we derive an estimate for the two-point function (2.1)

$$|\mathscr{W}^{(2)}(g_1, g_2)| \leqslant C \sum_{r=0}^\infty |d_r|^2 (2\pi)^{-r} 2^{r/2} H(L)\, m^{-L} (L-1)^{-r} \qquad \text{(A2.1)}$$

Thus we conclude that there are already convergent two point functions, if the test functions decrease faster than $p_0^{-L/2}$, $L > 1$. Let us introduce the symbol $\mathscr{K}_\lambda$ for the set of all functions $\tilde{g}$ satisfying

$$|p_0{}^L \tilde{g}(p)| \leqslant C, \qquad L > \lambda$$

$\mathscr{K}_\lambda$ is a dense subspace of $\mathscr{S}^{1/\lambda}$. (A2.1) shows that sufficient for the existence of $\mathscr{W}^{(2)}(g_1, g_2)$ for all $g_{1/2} \in \mathscr{K}_\lambda$, $\lambda > \frac{1}{2}$ is the condition

$$|d_r| \leqslant M(\lambda)^r, \qquad r > r_0 \tag{A2.2}$$

where $M(\lambda)$ is isoton in λ (more precisely $M(\lambda) = [2^{1/2}\pi(2\lambda - 1)]^{1/2}$). Equation (A2.2) is equivalent to $\rho\{d_r\} = 1$ and $\sigma\{d_r\} = M$ [cf. (2.9)]. If the test functions are taken out of $\mathscr{S}$, λ may be chosen arbitrarily large, but finite; i.e., sufficient for convergence in $\mathscr{S}'$ is $\rho\{d_r\} = 1$ and $\sigma\{d_r\}$ is finite. This is Jaffe's condition (Jaffe, 1965a).

If we restrict the test functions $\tilde{g}$ to the smaller spaces $\mathscr{S}_\alpha$, $0 < \alpha < \infty$, we have $H(L) \leqslant CA^L L^{L\alpha}$ (A1.1), (A1.3). We can diminish the factors beside $|d_r|^2$ in formula (A2.1), if we let L depend on r. An appropriate choice is

$$L(r) = \gamma r, \quad \gamma > 0$$

The aforementioned factors now decrease like $r^{(\gamma\alpha - 1)}$, γ arbitrarily small, greater zero. The sufficiency condition reads

$$|d_r| \leqslant r^{\delta r}, \qquad \delta < \tfrac{1}{2}, \qquad r > r_0$$

or

$$\rho\{d_r\} < 2$$

For $\tilde{g}_{1/2} \in \mathscr{D}$, i.e., $\alpha = 0$, the factors beside $|d_r|^2$ decrease arbitrarily quickly and there is no condition to be imposed on the series-coefficients d_r.

Analogously to Section 3, one can demonstrate that the stated conditions for the two-point function imply existence for the n-point functions.

We summarise the above results in the position space representation, introducing the spaces $\mathscr{K}^\lambda = \mathscr{F}\mathscr{K}_\lambda$.

Theorem A3

Sufficient for the convergence of $\chi(g) = \sum_{r=0}^\infty (d_r/r!){:}\phi^r{:}(g)$ in two dimensions for all test functions in

the space	is the condition
$\mathscr{K}^\lambda$, $\lambda \geqslant \frac{1}{2}$	$\rho\{d_r\} = 1$, $\sigma\{d_r\} = M(\lambda)$.
$\mathscr{S}(\alpha = \infty)$	$\rho\{d_r\} = 1$, $\sigma\{d_r\} < \infty$.
$\mathscr{S}^\alpha$, $0 < \alpha < \infty$	$\rho\{d_r\} < 2$.
$\mathscr{S}^0 = \mathscr{D}$	d_r is an arbitrary sequence.

The table of Theorem A3 shows that the spaces $\mathscr{S}^\alpha$ are not convenient for infinite series of increasing logarithmic singularities. All spaces $\mathscr{S}^\alpha$, $0 < \alpha < \infty$, are comprised in one condition which itself is, in higher space dimensions, the condition for localisability.

From Theorem A3, statements on the singularities on the light-cone and on the growth of the spectral functions can be derived. The spaces $\mathscr{K}^\lambda$ may be of use for a detailed discussion of the Thirring model. There, $M(\lambda)$ is proportional to the coupling constant.

Appendix 3

On the Precision of Condition α

In order to investigate the precision of condition α (cf. Theorem 1) the two-point function is smeared with a special function g_s. We now supplement the estimate leading from the last inequality before (2.12) to (2.12).

$$\Delta^r(g_s, g_s) \geqslant C_1 C_2^r \int \cdots \int \exp\left[-a\left(\sum_{l=1}^{r} p_l{}^0\right)^{1/\alpha}\right] \prod_{l=1}^{r} \frac{dp}{2p^0}$$

$$\geqslant C_1 C_2^r \int_m^\infty \cdots \int_m^\infty \exp\left[-a\left(\sum_{l=1}^{r} p_l{}^0\right)^{1/\alpha}\right] \prod_{l=1}^{r} (p_l{}^0 - m)^{\kappa-1}\, dp_l{}^0,$$

$$\kappa \geqslant 1$$

As in Section 2, the constants C_1, C_2 may be of different value in different lines. Passing to the variables $t_k = \sum_{l=1}^{k} p_l{}^0$, $1 \leqslant k \leqslant r$ gives

$$\Delta^r(g_s, g_s) \geqslant C_1 C_2^r \int_{rm}^{\infty} \int_{(r-1)m}^{t_r-m} \cdots \int_{m}^{t_2-m} \exp\left[-a t_r^{1/\alpha}\right] \times$$

$$\times \prod_{l=2}^{r} (t_l - t_{l-1} - m)^{\kappa-1} (t_1 - m)^{\kappa-1} \prod_{l=2}^{r} dt_l$$

Let us introduce the constants $\epsilon > 0$ and δ, so that $m/m + \epsilon < \delta < 1$. Then the inequality $l(m+\epsilon) \leqslant t_l$ implies

$$\frac{l-\delta}{l}\, t_l \leqslant t_l - m$$

For the t_{l-1} satisfying

$$(l-1)(m+\epsilon) \leqslant t_{l-1} \leqslant \frac{l-\delta}{l}\, t_l$$

holds

$$t_1 - m \geqslant t_1(\delta - m/m + \epsilon)$$

$$t_l - t_{l-1} - m \geqslant (t_l/l)(\delta - m/m + \epsilon)$$

Denoting $\delta - m/m + \epsilon = \mu$ we obtain

$$\Delta^r \geqslant C_1 C_2^r r!^{-(\kappa-1)} \mu^{(\kappa-1)r} \int_{r(m+\epsilon)}^{\infty} \int_{(r-1)(m+\epsilon)}^{[(r-\delta)/r]t_r} \cdots \int_{m+\epsilon}^{[(2-\delta)/2]t_2} \times$$

$$\times \exp[-a t_r^{1/\alpha}] \prod_{l=1}^{r} t_l^{\kappa-1} dt_l$$

Observing

$$\int_{(l-1)(m+\epsilon)}^{[(l-\delta)/l]t_l} t_{l-1}^{(l-1)\kappa-1} dt_{l-1} \geqslant \frac{t_l^{(l-1)\kappa}}{(l-1)\kappa} \left[\left(\frac{l-\delta}{l}\right)^{(l-1)\kappa} - \left(\frac{l-1}{l}\right)^{(l-1)\kappa} \right]$$

$$\geqslant \frac{t_l^{(l-1)\kappa}}{(l-1)\kappa} \left(\frac{l-1}{l}\right)^{(l-1)\kappa} (1-\delta)\kappa, \qquad l = 2, \ldots, r$$

$$\prod_{l=2}^{r} \left(\frac{l-1}{l}\right)^{(l-1)\kappa} \geqslant \exp[-\kappa(r-1)]$$

and passing to the variable $s = a t_r^{1/\alpha}$ one is led to (2.12)

$$\Delta^r(g_s, g_s) \geqslant C_1 C_2^r (r!)^{-\kappa} a^{-\kappa r \alpha} \int_{a[r(m+\epsilon)]^{1/\alpha}}^{\infty} \exp(-s) s^{\alpha\kappa r - 1} ds$$

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